MASS TRANSFER IN A TWO-LAYER POROUS MEDIUM

A. A. Roshal'

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Mass transfer in an actual porous medium, when no physicochemical interactions occur between the solution and the medium containing it, is characterized by macrodispersion taking place in the diffusion flow, as a result of the filtrational inhomogeneity of the medium [1, 2]. When the system parameters are related in a certain way, this process becomes predominant over the microdispersion due to disorderedness of the pore structure and comes to characterize almost entirely the distributions of the concentrations in space and time.

We consider the linear problem of mass transfer in a system of alternating layers of widely different permeabilities, in which the liquid movement occurs only along the most permeable layers parallel to the stratification. The hydrodynamic flow is assumed to be stationary and stabilized, and subject to Darcy's linear law.

Elementary analysis shows [1–3] that, in conditions of mass transfer in a filtration flow with layers lying above and below, and with values of the Peclet diffusion number such as occur in practice ($N_{Pe} \sim 10^3–10^6$), the curvature of the concentration distribution in the direction of the filtration flow movement can be neglected in comparison with the curvature of the concentration distribution in the transverse direction, i. e., transfer in the permeable layers occurs primarily by forced convection, and in the relatively impermeable layers, as a result of molecular diffusion in the free pore space.

In accordance with the above assumptions, the equations of mass transfer for the permeable and relatively impermeable layers may be written, respectively, as

$$n_0 \frac{\partial C_0}{\partial t} = \frac{1}{m_0}\left[D_+ \frac{\partial C_+}{\partial x_2}\bigg|_{x_2=x_{2+}} + D_- \frac{\partial C_-}{\partial x_2}\bigg|_{x_2=x_{2-}}\right] - v_0 \frac{\partial C_0}{\partial x_1} \tag{0.1}$$

$$n_{+-} \frac{\partial C_{+-}}{\partial t} = D_{+-} \frac{\partial^2 C_{+-}}{\partial x_2^2} \tag{0.2}$$

Here, n_{+-}, D_{+-} are the effective porosity and coefficient of molecular diffusion in the relatively impermeable layers lying, respectively, above or below the permeable layer in question, the ordinates of the corresponding boundaries being, respectively, x_{2+} and x_{2-} ; m_0 and n_0 are the thickness and effective porosity of the permeable layer; v_0 is the velocity of the filtration flow, moving in the direction of the x_1 coordinate (see the figure); C_0 and C_{+-} are the relative concentrations of the solute in the permeable and relatively impermeable layers; t is current time; and n_0 and C_0 are the means obtained by integration over the interval $[x_{2-}; x_{2+}]$.

To complete the statement of the problem, suitable boundary conditions have to be found for systems (0.1) and (0.2), in which the numbers of equations correspond to the numbers of the two types of layer; in particular, a boundary condition of the fourth kind is assumed to be satisfied on the boundary of the filtration flow with a relatively impermeable layer.

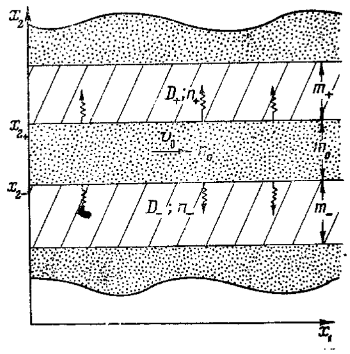
1. Isolated two-layer medium. Consider an isolated system consisting of two horizontal layers of essentially different permeabilities such that our above assumptions remain valid. In other words, we confine ourselves here to solution of the "interior" problem; this will enable us to describe the process most fully in the "pure form," independently of the external conditions.

After imposing "impermeability" conditions on the upper and lower boundaries of the system and introducing the generalized independent variables

$$\eta = \frac{D_+ x_1}{m_0 m_+ v_0}, \quad \tau = \frac{D_+ n_0}{n_+ m_+^2 v_0}\left(\frac{v_0}{n_0} t - x_1\right), \quad y = \frac{x_2 - x_{2+}}{m_+} \tag{1.1}$$

Eqs. (0.1) and (0.2) for the permeable and relatively impermeable layers may be written, respectively, as

$$\frac{\partial C_0}{\partial \eta} - \frac{\partial C_+}{\partial y}\bigg|_{y=0} = 0, \quad \frac{\partial C_+}{\partial \tau} - \frac{\partial^2 C_+}{\partial y^2} = 0 \tag{1.2}$$


x_2
$D_+; n_+$
m_+
x_{2+}
U_0
n_0
m_0
x_{2-}
$D_-; n_-$
m_-
x_1

Here m_+ is the thickness of the relatively impermeable layer.

The system of equations (1.2) is completed by means of the conditions

$$C(y, \eta, 0) = 0, \quad \frac{\partial C(1, \eta, \tau)}{\partial y} = 0, \quad C(y, 0, \tau) = 1 \tag{1.3}$$

Laplace-transforming system (1.2) and using the zero initial conditions (the first of Eqs. (1.3)), we get

$$\frac{\partial U_0}{\partial \eta} = \frac{\partial U_+}{\partial y}\bigg|_{y=0}, \quad \frac{\partial^2 U_+}{\partial y^2} - pU_+ = 0 \tag{1.4}$$

Here,

$$U(\eta, p) = L\{C(\eta, \tau)\} = p\int_0^\infty e^{-p\tau} C(\eta, \tau)\, d\tau$$

by the Carson-Heaviside transformation.

The general integral of the second of Eqs. (1.4) can be written as

$$U_+ = A\operatorname{sh}(\sqrt{p}y) + B\operatorname{ch}(\sqrt{p}y) \tag{1.5}$$

Hence, using the second of conditions (1.3), the first of Eqs. (1.4) reduces to the form

$$\frac{dU_0}{d\eta} + U_0\sqrt{p}\operatorname{th}\sqrt{p} = 0 \tag{1.6}$$

Finally, using the third of conditions (1.3), the solution of the problem for the permeable layer in the image space is found to be

$$U_0(\eta, p) = \exp\,[-\eta\sqrt{p}\operatorname{th}\sqrt{p}] \tag{1.7}$$

The integral form of the function whose Laplace transform is (1.7) can be found from Mellin's inversion formula

$$C_0(\eta, \tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \exp[-\eta\sqrt{p}\operatorname{th}\sqrt{p} + p\tau]\,\frac{dp}{p} \tag{1.8}$$

It is easily shown that the integrand of (1.8) is single-valued throughout the region of the complex variable p, has a simple pole at p = 0 with residue unity, and has an infinity of essential singularities p_k, located on the negative real axis:

$$p_k = -\,{}^1\!/_4\pi^2(2k+1)^2 \quad (k = 0, 1, 2, \ldots)$$

In addition, the integrand satisfies Jordan's lemma, whence, applying the residue theorem, (1.8) can be written as a series in θ-functions of a zero argument and their derivatives with respect to some parameter (see [4] for the relevant definitions).

A more convenient integral form, from the point of view of numerical computations, is

$$C_0(\eta, \tau) = 0.5 + \frac{2}{\pi}\int_0^\infty \exp\left[-\frac{1}{2}\eta z\,\frac{\operatorname{sh} z - \sin z}{\operatorname{ch} z + \cos z}\right]$$
$$\times \sin\left[\frac{1}{2}\tau z^2 - \frac{1}{2}\eta z\,\frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z}\right]\frac{dz}{z} \tag{1.9}$$

which may be obtained from (1.8) by a series of elementary transformations.

Next, using (1.5) and the second of conditions (1.3), we obtain the expression

$$U_+(y, \eta, p) = U_0(\eta, p)\,\frac{\operatorname{ch}(1-y)\sqrt{p}}{\operatorname{ch}\sqrt{p}} \tag{1.10}$$

for the image of the concentration in the relatively impermeable layer.

Hence

$$C_+(y, \eta, \tau) = -\int_0^\tau C_0(\eta, \tau - \varepsilon) \frac{\partial \vartheta_2(^1/_2 y; \varepsilon)}{\partial y} d\varepsilon \tag{1.11}$$

Here and below, ϑ_l $(l = 0, 1, 2, 3)$ denote the θ-functions.

Using (1.10) and (1.11), we can write the diffusion flow in the relatively impermeable layer as

$$\begin{aligned} q_+ &= -D_+ \frac{\partial C_+}{\partial x_2}\bigg|_{x_2 = x_{2+}} = -\frac{D_+}{m_+}\frac{\partial C_+}{\partial y}\bigg|_{y=0} = \frac{D_+}{m_+}\int_0^\tau C_0(\eta, \tau - \varepsilon) \frac{\partial^2 \vartheta_2(0; \varepsilon)}{\partial y^2} d\varepsilon \\ &= \frac{D_+}{m_+}\frac{d}{d\tau}\int_0^\tau C_0(\eta, \tau - \varepsilon)\, \vartheta_2(0; \varepsilon)\, d\varepsilon \end{aligned} \tag{1.12}$$

In addition, starting from the first of Eqs. (1.2) and (1.9), the diffusion flow in the relatively impermeable layer is

$$\begin{aligned} q_+ = \frac{D_+}{m_+}\frac{\partial C_0}{\partial \eta} &= \frac{D_+}{m_+}\frac{1}{\pi}\int_0^\infty \exp\left[-\frac{1}{2}\eta z \frac{\operatorname{sh} z - \sin z}{\operatorname{ch} z + \cos z}\right] \\ &\times \left\{\frac{\operatorname{sh} z - \sin z}{\operatorname{ch} z + \cos z} \sin\left[\frac{1}{2}\tau z^2 - \frac{1}{2}\eta z \frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z}\right]\right. \\ &\left. + \frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z} \cos\left[\frac{1}{2}\tau z^2 - \frac{1}{2}\eta z \frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z}\right]\right\} dz \end{aligned} \tag{1.13}$$

From (1.10), the image of the integral-mean value of the concentration in the relatively impermeable layer is

$$U_+^*(\eta, p) = \int_0^1 U_+(y, \eta, p)\, dy = U_0(\eta, p) \frac{1}{\sqrt{p}} \operatorname{th} \sqrt{p} \tag{1.14}$$

The corresponding inverse transform can be written as

$$C_+^*(\eta, \tau) = \int_0^1 C_+(y, \eta, \tau)\, dy = \int_0^\tau C_0(\eta, \tau - \varepsilon)\, \vartheta_2(0; \varepsilon)\, d\varepsilon \tag{1.15}$$

or, recalling (1.6) and (1.9),

$$\begin{aligned} C_+^*(\eta, \tau) = -\int_0^\tau \frac{\partial C_0}{\partial \eta} d\varepsilon &= \frac{2}{\pi}\int_0^\infty \exp\left[\frac{1}{2}\eta z \frac{\operatorname{sh} z - \sin z}{\operatorname{ch} z + \cos z}\right] \\ &\times \left\{\frac{\operatorname{sh} z - \sin z}{\operatorname{ch} z + \cos z} \sin\left[\frac{1}{2}\tau z^2 - \frac{1}{2}\eta \tau \frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z}\right]\right. \\ &\left. - \frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z} \cos\left[\frac{1}{2}\tau z^2 - \frac{1}{2}\eta z \frac{\operatorname{sh} z + \sin z}{\operatorname{ch} z + \cos z}\right]\right\} \frac{dz}{z^2} \end{aligned} \tag{1.16}$$

2. "Open" two-layer medium. Let us consider in general terms the solution of the analogous problem for a two-layer medium when an ideally "wettable" medium lies beyond the outer boundary of the relatively impermeable layer, so that an open system is obtained. In other words, the concentration at the outer boundary of the two-layer system is identical with the concentration of the surrounding medium, which, in turn, is assumed to be constant in space and time. The boundary condition for the outer boundary (1.3) now becomes

$$C(1, \eta, \tau) = 0 \tag{2.1}$$

whence, using (1.5) and the first of Eqs. (1.4), the image of the concentration in the permeable layer is

$$U_0(\eta, p) = \exp[-\eta\sqrt{p} \operatorname{cth}\sqrt{p}] \tag{2.2}$$

The inverse of the Laplace transform (2.2) can be written in a form similar to (1.9). Typically, the limiting solution of our problem is

$$\lim_{\tau \to \infty} C_0 = e^{-\eta} \tag{2.3}$$

Further, using (1.5) and condition (2.1), we obtain the image

$$U_+(y, \eta, p) = U_0(\eta, p) \frac{\operatorname{sh}(1-y)\sqrt{p}}{\operatorname{sh}\sqrt{p}} \tag{2.4}$$

Hence the concentration in the relatively impermeable layer is

$$C_+(y, \eta, \tau) = -\int_0^\tau C_0(\eta, \tau-\varepsilon) \frac{\partial \vartheta_3(^1/_2 y; \varepsilon)}{\partial y} d\varepsilon \tag{2.5}$$

The image of the diffusion flow through the outer boundary of the relatively impermeable layer is

$$L\{q_{+-}\} = -D_- \frac{\partial U_+}{\partial x_2}\bigg|_{x_2=x_{2+}+m_+} = -\frac{D_+}{m_+}\frac{\partial U_+}{\partial y}\bigg|_{y=1} = -\frac{D_+}{m_+} U_0(\eta, p) \frac{\sqrt{p}}{\operatorname{sh}\sqrt{p}} \tag{2.6}$$

The inverse of (2.6) is

$$q_{+-} = \frac{D_+}{m_+}\frac{\partial}{\partial x}\int_0^\tau C_0(\eta, \tau-\varepsilon)\,\vartheta_0(0; \varepsilon)\,d\varepsilon = \frac{D_+}{m_+}\int_0^\tau C_0(\eta, \tau-\varepsilon) \frac{\partial^2 \vartheta_3(^1/_2; \varepsilon)}{\partial y^2} d\varepsilon \tag{2.7}$$

The diffusion flow in the relatively impermeable layer is

$$\begin{aligned} q_{++} &= \frac{D_+}{m_+} L^{-1}\left\{\frac{1}{\sqrt{p}} \operatorname{cth}\sqrt{p}\, U_0(\eta, p)\right\} = \frac{D_+}{m_+}\frac{\partial}{\partial \tau}\int_0^\tau C_0(\eta, \tau-\varepsilon)\,\vartheta_0\left(\frac{1}{2}; \varepsilon\right) d\varepsilon \\ &= \frac{D_+}{m_+}\int_0^\tau C_0(\eta, \tau-\varepsilon) \frac{\partial^2 \vartheta_3(0; \varepsilon)}{\partial y^2} d\varepsilon \end{aligned} \tag{2.8}$$

The image of the integral-mean value of the concentration in the relatively impermeable layer may be written as

$$U_+^*(\eta, p) = \int_0^1 U_+(y, \eta, p)\,dy = -U_0(\eta, p)\left[\operatorname{cth}\sqrt{p} + \frac{1}{\operatorname{sh}\sqrt{p}}\right]\sqrt{p} \tag{2.9}$$

whence the original is

$$C_+^*(\eta, \tau) = -\int_0^\tau C_0(\eta, \tau-\varepsilon)\left[\vartheta_0\left(\frac{1}{2}; \varepsilon\right) - \vartheta_0(0; \varepsilon)\right] d\varepsilon \tag{2.10}$$

3. Asymptotic expansions. It is easily seen, from image (1.7), that the asymptotic form of the solution when τ and η are small is

$$C_0(\eta, \tau) \approx L^{-1}\{\exp[-\eta\sqrt{p}]\} = \operatorname{erfc}\left[\frac{\eta}{2\sqrt{\tau}}\right] \tag{3.1}$$

It should be mentioned that solution (3.1) was obtained by Lauwerier [3] for the case of an infinitely thick, relatively impermeable layer.

The concentration in the relatively impermeable layer and the diffusion flow in this layer are then, respectively,

$$C_+(y, \eta, \tau) \approx \operatorname{erfc}\left[\frac{\eta+y}{2\sqrt{\tau}}\right] \tag{3.2}$$

$$q_+ \approx -\frac{D_+}{m_+\sqrt{\pi\tau}} \exp\left[-\frac{\eta^2}{4\tau}\right] \tag{3.3}$$

These asymptotic forms are virtually always applicable (within an error of 5%) when $\tau < 0.25$, regardless of the values of η and y. Expansions (3.1) and (3.3) are also applicable for arbitrary values of τ when $\eta < 0.1$, and expansion (3.2) when $\eta + y < 0.1$. But an asymptotic expansion can be written for the concentration in the relatively impermeable layer, for which no extra restriction on y is required. For this, expansion (3.1) is simply substituted into (1.11).

A second asymptotic expansion of the solution may be obtained from the following considerations. Recalling the general theory of meromorphic functions, image (1.7) may be written as

$$U_0(\eta, p) = \exp\left[-\eta\frac{2p}{p+2}\sum_{k=0}^{\infty}\frac{p+2}{p+{}^1/_4\pi^2(2k+1)^2}\right] \tag{3.4}$$

The series in (3.4) is uniformly convergent to unity in a circle of radius |p| = 2. Hence the asymptotic expansion of image (3.4) may be written as

$$U_0(\eta, p) \approx \exp\left[-\eta\frac{2p}{p+2}\right] \tag{3.5}$$

Further,

$$L^{-1}\left\{\frac{2p}{p+2}\exp\left[-2\eta\frac{p}{p+2}\right]\right\} = 2\exp\left[-(2\eta+2\tau)\right] I_0(4\sqrt{\eta\tau}) \tag{3.6}$$

where $I_0(x)$ is the zero-order modified Bessel function of the first kind; the solution can be written as

$$C_0(\eta, \tau) \approx J(2\eta, 2\tau) \tag{3.7}$$

The function

$$J(\sigma, \varkappa) = 1 - e^{-\varkappa}\int_0^\sigma e^{-z} I_0(2\sqrt{\varkappa z})\, dz \tag{3.8}$$

was obtained by Arzelius [5] and Schumann [6] and was very fully investigated by Goldstein [7–11].

It can be shown that solution (3.7) is obtainable directly from the system of equations (1.2) by putting

$$\left.\frac{\partial C_+}{\partial y}\right|_{y=0} = 2(C_+^* - C_0) \tag{3.9}$$

where C_+^* is the value of the concentration in the central section of the relatively impermeable layer. Equation (3.9) presupposes a linear law of mass transfer, analogous to Newton's law, at the boundary, and in addition, a condition that the relatively impermeable layer have infinite diffusion conductivity must be introduced into the initial system. In other words, the initial system (1.2) must take the form

$$\frac{\partial C_0}{\partial \eta} = 2(C_+^* - C_0), \quad \frac{\partial C_+^*}{\partial \tau} = 2(C_0 - C_+^*) \tag{3.10}$$

The following expressions for the integral-mean values of the concentration in the relatively impermeable layer and the diffusion flow in this layer may be obtained from the image (3.5):

$$C_+'(\eta, \tau) \approx C_+^*(\eta, \tau) = 2\int_0^{2\tau} e^{-2(\eta+z)} I_0(4\sqrt{\eta z})\, dz$$

$$= J(2\eta, 2\tau) - e^{-2(\eta+\tau)} I_0(4\sqrt{\eta\tau}) \tag{3.11}$$

$$q_+ \approx -\frac{2D_+}{m_+} e^{-2(\eta+\tau)} I_0(4\sqrt{\eta\tau}) \tag{3.12}$$

Numerical analysis shows that the asymptotic forms (3.7), (3.11), and (3.12) can be used in practice (within an error of 3%) when $\tau > \eta > 0.2$.

It may be seen from the Laplace image (2.2) that the asymptotic forms of the solution in the problem of an open two-layer medium in which the parameters η and τ have small values are the same as the corresponding forms (3.1), (3.2), and (3.3) for an isolated two-layer medium. These solutions are practically useful (within an error of 5%) when $\tau < 0.15$ and η and y have arbitrary values, and also when $\eta + y > 0.05$ and τ has arbitrary values.

The second asymptotic forms of the solutions for an open two-layer medium may be obtained by a similar procedure to that used for an isolated two-layer medium. Let us write image (2.2) as

$$U_0(\eta, p) = \exp\left[-\eta - \eta\frac{2p}{p+6}\sum_{k=0}^{\infty}\frac{p+6}{p+k^2\pi^2}\right] \tag{3.13}$$

The series in (3.13) is uniformly convergent to unity in a circle of radius |p| = 6. Hence the asymptotic form of the concentration in the permeable layer of the open system is

$$C_0(\eta, \tau) \approx L^{-1}\left\{\exp\left[-\eta - \eta\frac{2p}{p+6}\right]\right\} = e^{-\eta} J(2\eta, 6\tau) \tag{3.14}$$

The remaining asymptotic forms of the solutions in the open system problem may be found similarly.

4. Some generalizations. When the parameters η and τ have small values and no restriction is imposed on the capacities of the relatively impermeable layers, the solutions of the problem for strongly differentiated systems are the same as (3.1). This makes it possible for mass transfer interaction between the different permeable layers via the relatively impermeable layers to be ignored in practice under certain conditions.

Our scheme of stratified inhomogeneities should also degenerate with large values of η and τ. It may be assumed that the process can then be described by a diffusion equation with a convective term:

$$N\frac{\partial C}{\partial t}+V\frac{\partial C}{\partial x}-D^*\frac{\partial^2 C}{\partial x^2}=0 \tag{4.1}$$

Here, the effective porosity and effective filtering speed in the quasi-homogeneous layer are, respectively,

$$N=\frac{m_+n_++m_0n_0}{m_++m_0},\quad V=\frac{v_0m_0}{m_++m_0} \tag{4.2}$$

The coefficient D* in (4.1) is to be regarded as an effective diffusion coefficient, while the process itself might be termed mass-transfer diffusion, by analogy with the phenomenon of convective diffusion.

Let us prove our assumption by writing the asymptotic form of image (3.5) as

$$U_0(\eta,\lambda)\approx\exp\left[\left({}^1\!/_2-\sqrt{{}^1\!/_4+(a+1)\lambda}\right)\eta\right] \tag{4.3}$$

Here,

$$U_0(\eta,\lambda)=L^{-1}\{C_0(\eta,\omega)\},\quad a=\frac{m_0n_0}{m_+n_+},\quad \omega=\frac{D_0t}{m_+{}^2n_+}$$

The inverse of image (4.3) may be written as

$$C_0(\eta,\omega)\approx 0{,}5\left\{\operatorname{erfc}\left[\frac{(a+1)\eta-\omega}{2\sqrt{(a+1)\omega}}\right]+e^{\eta}\operatorname{erfc}\left[\frac{(a+1)\eta+\omega}{2\sqrt{(a+1)\omega}}\right]\right\} \tag{4.4}$$

The asymptotic form (4.4) of solution (3.7) may be used (within an error of 3%) when $\tau > \eta > 0.1$ (or $\omega > (a+1)\eta$; $\eta > 0.1$), i. e., in practice it can be used whenever expansion (3.7) is applicable.

Expression (4.4) is the solution of the diffusion problem (4.1) under the first and third of boundary conditions (1.3) if we put

$$D^*=\frac{m_+m_0{}^2v_0{}^2}{D_+(m_0+m_+)} \tag{4.5}$$

Let us dwell in more detail on the conditions under which expansions (3.7) and (4.4) can be used. To this end, consider the physical significance of the inequality $\eta < \tau$. Using (1.1) and a number of transformations, the condition under which (3.7) and (4.4) can be used may be written as

$$(m_0n_0+m_+n_+)x_1 < m_0v_0t \tag{4.6}$$

Hence the initiation of a quasi-stationary mode of mass transfer, characterized by the mass-exchange law (3.9), is defined by the instant when the volume of filtered liquid passing through a fixed unit cross section x_1 exceeds the total free capacity of the system. It was pointed out in [12] that, in the present terminology, this condition corresponds to the instant when the sum of the integral-mean instantaneous values of the concentrations in the permeable and relatively impermeable layers is equal to unity.

However, (4.6) is not a sufficient condition. The entry part of the flow must be characterized by a sharp deformation of the diffusion flow in the relatively impermeable layer, caused by the high intensity of the mass-release process. The relative magnitude of this part is uniquely determined by the condition for applicability of solutions (3.7) and (4.4) when $\eta > 0.2$. Recalling (1.1), this condition is found to be

$$\frac{x_1}{m_0}>0.2\frac{m_+v_0}{D_+}=0.2N_{\mathrm{Pe}} \tag{4.7}$$

where N_{Pe} is the Peclet diffusion number, based on the characteristic dimension, namely, the thickness of the relatively impermeable layer, and the coefficient of molecular diffusion in this layer.

A complete analogy appears here between the phenomenon discussed and the sharp restructuring that takes place when a stable concentration profile develops in the liquid itself in the entry part close to the boundary of the considered region. In addition, these parts are comparable in size, though the length of section (4.7) is actually rather too high, owing to the fact that under our assumptions concentration profile deformation is completely ignored.

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